Neutron Transport Equation

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Overview

- Basic physical assumptions
- Transport phenomena
- Collision model
- Basic physical quantities
- Stochastic or deterministic
- Equation for neutron movements
- Approximations, simplifications etc.
Basic Physical Assumptions

- Neutrons are dimensionless points
- Neutron – neutron interactions are neglected
- Neutrons travel in straight lines
- Collisions are instantaneous and point-like
- Background material properties are isotropic
- Properties of background material are known and time-independent
Neutrons as Waves

\[ \lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}} \] - de Broglie relation

\[ \lambda_{(\text{Å})} = \frac{0.287}{\sqrt{E_{(eV)}}} \]
\[ \lambda_{(\text{Å})} = \frac{4 \times 10^3}{v_{(m/s)}} \]

Ångström [Å] = 10^{-10} m = 0.1 nm
Examples of Neutron Waves

Example: \( E = 0.025 \text{ eV} \quad \nu = 2.2 \times 10^3 \text{ m/s} \quad \lambda = 1.8\text{Å} \ll \text{mfp} \)

\( E = 1 \text{ MeV} \quad \nu = 1.4 \times 10^7 \text{ m/s} \quad \lambda = 0.000287 \text{Å} \)

\( E = 1000 \text{ MeV} \quad \nu \sim 3 \times 10^8 \text{ m/s} \quad \lambda = 0.9 \times 10^{-5} \text{Å} \)

Size of atom \( \sim 1 \text{Å} \)

Size of nucleus \( \sim 10^{-4} \text{Å} \)

Distance between atoms in solids \( \sim \text{few Å} \)
Microscopic X-Sections

\[ \sigma \text{ (barns)} \]

Energy (eV)

- fission
- capture

\[ ^{239}\text{Pu} \]
Two Kinds of Transport

Random walk (self-diffusion)  Collective transport

Examples:

(a) Neutrons in reactor
(b) Gas molecules

Equation:

(a) Neutrons do not collide with each other
(b) Molecules do collide with each other

Examples: Neutrons in reactor  Gas molecules

Equation: NTE  Boltzmann

\[ n \sim 10^8 \]
\[ N_B \sim 10^{22} \]
Basic Physical Quantities

Neutron density \( n(r,t) \) \( n(r,E,t) \) \( N(r,\Omega,E,t) = N(r,v,t) \)

Reaction rate \( R_x(r,E,t) = \Sigma_x(r,E) \nu n(r,E,t) \)

Neutron flux \( \phi(r,E,t) \equiv \nu n(r,E,t) \)

\[
P(t) = \varepsilon_f \int_0^\infty \int_V \Sigma_f(r,E) \phi(r,E,t) d^3rdE
\]

There is no equation in terms of \( n(r,t) \) or even \( n(r,E,t) \)!

We have to consider direction \( N(r,\Omega,E,t) \) or \( N(r,v,t) \)
Stochastic Nature of Physical Variables

We can speak only of average or expected value $N(r, \Omega, E, t)$

Neutron Transport Model
(List of physical processes)

Probabilities of all processes
Equation for probabilities
Equation for expected neutron density

Lenard Pal, 1958

$P_{N_1}(R_1, t_1 | r, v, t)$
$P_{N_1, N_2}(R_1, t_1, R_2, t_2 | r, v, t)$
$P_{N_1, N_2, N_3}(R_1, t_1, R_2, t_2, R_3, t_3 | r, v, t)$
Physical Units

Number of neutrons expected in $d^3 r$

$$n(r, t)d^3 r \longrightarrow \left[ n(r, t) \right] = \frac{n}{\text{cm}^3}$$

Number of neutrons expected in $d^3 r dE$

$$n(r, E, t)d^3 r dE \longrightarrow \left[ n(r, E, t) \right] = \frac{n}{\text{cm}^3 \cdot \text{J}}$$

Number of neutrons expected in $d^3 r d\Omega dE$

$$n(r, \Omega, E, t)d^3 r d\Omega dE \longrightarrow \left[ n(r, \Omega, E, t) \right] = \frac{n}{\text{cm}^3 \cdot \text{sr} \cdot \text{J}}$$
Spatial and Angular Variables

\[ N(r, \Omega, E, t) \]

\[ N(r, \Omega, E, t)d^3r d\Omega dE \]

\[ \frac{dr}{dt} = \mathbf{v} = v\Omega \]

\[ E = \frac{mv^2}{2} \]
Neutron Angular Distributions

\[ \Phi(r, \Omega, E, t) \equiv \nu N(r, \Omega, E, t) \]

\[ \phi(r, E, t) \equiv \int_0^{2\pi} \int_0^{\pi} \Phi(r, \Omega, E, t) \sin(\theta) \, d\Omega \Rightarrow R_x(r, E, t) = \Sigma_x(r, E) \phi(r, E, t) \]

\[ \phi(r, t) \equiv \int_0^{\infty} \phi(r, E, t) \, dE \Rightarrow R_x(r, t) = \Sigma_x(r) \phi(r, t) \]
Leonhard Euler's (1707-1783) description:

We fix a small volume \(\Rightarrow \frac{\partial N}{\partial t}\)

Joseph Lagrange's (1736-1813) description:

We let a small volume move \(\Rightarrow \frac{dN}{dt}\)
Substantial Derivative

Along streamlines:
\[ \mathbf{r} = \mathbf{r}(t); \quad \mathbf{v} = \mathbf{v}(t) \]
\[ N(\mathbf{r}, \mathbf{v}, t) = N(\mathbf{r}(t), \mathbf{v}(t), t) \]

Gravity, \( \mathbf{F} \), is negligible for neutrons

Gradient in physical space

\[ \frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \frac{\partial N}{\partial \mathbf{r}} + \frac{d\mathbf{v}}{dt} \cdot \frac{\partial N}{\partial \mathbf{v}} = \]
\[ = \frac{\partial N}{\partial t} + \mathbf{v} \cdot \nabla N + \frac{\mathbf{F}}{m} \cdot \nabla \cdot N \]

Gradient in velocity space
Derivation of NTE

In the moving volume:

\[ \frac{dN}{dt} = (\text{Gain}) - (\text{Loss}) \]

(Loss) = \[ \Sigma_t (r, E) vN (r, \Omega, E, t) \]

\[ \sigma_s (\Omega', E' \rightarrow \Omega, E) = \left[ \frac{\text{cm}^2}{\text{sr} \cdot \text{eV}} \right] \]

\[ \Sigma_s (\Omega', E' \rightarrow \Omega, E) = \sigma_s (\Omega', E' \rightarrow \Omega, E) N_B \]

(Gain) = \[ \int_0^\infty \int_{4\pi} \Sigma_s (r, \Omega', E' \rightarrow \Omega, E) v'N(r, \Omega', E', t) d\Omega' dE' + Q \]
Neutron Transport Equation

\[
\frac{dN}{dt} = \frac{\partial N(\mathbf{r}, \Omega, E, t)}{\partial t} + \mathbf{v} \cdot \nabla N = \\
= -\Sigma_t \nu N + \int_0^{4\pi} \int_0^\infty \Sigma_s (\mathbf{r}, \Omega', E' \rightarrow \Omega, E) \nu' N(\mathbf{r}, \Omega', E', t) d\Omega' dE' + Q
\]

\[
\begin{align*}
N(\mathbf{r}, \Omega, E, t = 0) &= N_0(\mathbf{r}, \Omega, E) & : \text{initial condition} \\
N(\mathbf{r}_s, \Omega, E, t)|_{\Omega \cdot \mathbf{n}_s < 0} &= 0 & : \text{boundary (free surface) condition}
\end{align*}
\]
Collision Model

Collisions are point-like and instantaneous
NTE in Terms of Angular Flux

\[
\frac{1}{v} \frac{\partial \Phi(r, \Omega, E, t)}{\partial t} = -\Omega \cdot \nabla \Phi - \Sigma_t \Phi + Q + \\
+ \int_0^\infty \int_0^{4\pi} \Sigma_s(r, \Omega', E' \rightarrow \Omega, E) \Phi(r, \Omega', E', t) \, d\Omega' \, dE'
\]
Possible Approaches

To handle general case:
- NTE
- Monte Carlo

To handle energy:
- One group (speed)
- Two-group model
- Multi-group Model

To handle angle:
- Diffusion model
- $S_N$ method
- Spherical harmonics

Fermi age theory = Diffusion + Slowing-down

To handle both space and angle:
- Continues slowing-down
- Discontinues slowing-down
Treatment of Angular Variable

1) Series expansion: \( F(x) = a_0 + a_1 x + \ldots = a_0/2 + a_1 \cos x + b_1 \sin x + \ldots \)

\[
\Phi(r, \Omega, E, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_l^m(r, E, t) Y_l^m(\Omega) = \frac{1}{4\pi} \phi(r, E, t) + \frac{3}{4\pi} \Omega \cdot J(r, E, t) + \ldots
\]

Spherical harmonics

\[
\phi(r, t) \equiv \int_{4\pi} \Phi(r, \Omega, t) d\Omega; \quad J(r, t) \equiv \int_{4\pi} \Omega \Phi(r, \Omega, t) d\Omega
\]

2) Discrete ordinates:
Treatment of Energy Variable

One-group

Two-group

Increasing energy, $E$

Multi-group

Group 1

Group 2

Group $G$ -1

Group $G$
Reactor Models

- Diffusion
- Homogeneous
- Quasi-Homogeneous
- Non-Homogeneous (Heterogeneous)
- One group (speed)
- Two group
- Multi group
Geometric Simplifications
Simplest Model

Energy: One-group

Space: 1-Dim

Angle: No angle = Isotropic = Diffusion
The END